

# On the mass spectrum of noncommutative Schwinger model in Euclidean $\mathbb{R}^2$ space

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The mass spectrum of the noncommutative QED in two-dimensional Euclidean  $\mathbb{R}^2$  space is derived first in a perturbative approach at one-loop level and then in a nonperturbative approach using the equivalent bosonized noncommutative effective action. It turns out that the mass spectrum of noncommutative QED in two dimensions reduces to a single non-interacting meson with mass  $M_\gamma = \frac{g}{\sqrt{\pi}}$ , as in commutative Schwinger model.

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## I. INTRODUCTION

Noncommutative field theories [1, 2] are in general characterized by the replacement of familiar product of functions with noncommutative Moyal star-product [3], as the simplest realization of the canonical commutation relation between space-time coordinates,  $[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}$ . Their deformed Lagrangian densities can therefore be written as an infinite series in terms of higher order temporal and spatial derivative of functions. From this point of view, noncommutative field theories can be understood as a class of nonlocal higher-derivative field theories, which appear in other areas of physics too. As it is argued in [4], unconstrained, nonlocal higher-derivative theories, having more degrees of freedom than lower-derivative theories, are dramatically different from their lower-derivative counterparts. Nonlocal field theories appear, in general, as effective field theories in a low-energy limit of a larger theory. In particular, space-space noncommutative field theories describe the low energy effective field theories of string theory in a background magnetic field. In [5], space-time noncommutative field theory is searched by considering open strings in a constant background electric field. It is shown that here, in contrast to the magnetic case, a critical electric field exists beyond which the theory does not

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make sense, and that this critical field prevents us from finding a limit in which the theory becomes a field theory on a noncommutative space-time. In so far, string theory does not admit a space-time noncommutative quantum field theory, as its low energy limit, with the exception of light-like noncommutativity [6]. Whereas space-space noncommutative field theories suffer from a mixing of ultraviolet and infrared singularities in their perturbative dynamics [7], space-time noncommutative theories in Minkowski space seem to be seriously acausal and inconsistent with conventional Hamiltonian evolution [5]. Besides their  $S$ -matrix fail to be unitary [8]. In [9] attempts are made to quantize theories with space-time noncommutativity, that leads, however, to inconsistencies. A path integral formulation of space-time noncommutativity using Schwinger's action principle is introduced in [10], from which the canonical structure for higher derivative theory is recovered. Using a simple field theory non-local in time, it is shown that the quantization on the basis of a naive interaction picture is not justified if the interaction contains non-local terms in time. It is shown that a unitary  $S$ -matrix can only be defined by using a modified time ordering, but the positive energy condition is spoiled together with a smooth Wick rotation to Euclidean theory. In [11], noncommutative QED in a 1+1 dimensional Minkowski space is expanded in the order of the noncommutativity parameter  $\theta$  up to  $\mathcal{O}(\theta^3)$ . The resulting theory is non-local in time and can be considered as a higher derivative theory including only a finite number of time derivatives. Using the method of perturbative quantization, it is then perturbatively quantized up to  $\mathcal{O}(e^2, \theta^3)$ , where  $e$  is the corresponding QED coupling constant. Recently, in [12], space-time noncommutative field theories are examined in various pictures and quantization procedures. It is shown that different quantization procedures lead to inconsistencies when time is taken as noncommutative coordinate. The results in [12] are consistent with the results in [5, 6, 8, 10] and with string theory which, as we have mentioned before, “does not admit a space-time noncommutative quantum field theory, as its low energy limit, with the exception of light-like noncommutativity” [6].

In this paper, we consider noncommutative QED in two-dimensional *Euclidean space* (noncommutative QED<sub>2</sub>), where the Minkowski time  $x_0 = t$  is replaced by the Euclidean imaginary time  $x_2 \equiv it$ , and the canonical commutation relation between the coordinates is reduced to  $[x_i, x_j] = i\theta\epsilon_{ij}$ , with  $i, j = 1, 2$  and  $\epsilon_{ij}$  the two-dimensional antisymmetric tensor. Hence, in contrast to its Minkowskian formulation, in the Euclidean space the noncommutativity is defined between two “equivalent” spatial coordinates, and the noncommutative field theory based on this formulation is therefore free of the before mentioned problems of 1 + 1 dimensional noncommutative field theories.

Having a Euclidean formulation of noncommutative QED<sub>2</sub>, we are in particular interested in the mass spectrum of the theory, that plays the rôle of the noncommutative counterpart of the well-known Schwinger model [13]. Originally introduced as a toy model to understand the confinement properties

of strong interaction, the Euclidean formulation of the Schwinger model turns out to have many applications in system where the physics is effectively reduced to two spatial dimensions. One example of such a system is  $3+1$  dimensional QED in the presence of constant magnetic fields. For strong enough magnetic fields, aligned in the third spatial direction, it is known that, because of the phenomenon of magnetic catalysis [14], the system is reduced to a two-dimensional surface perpendicular to the direction of the external magnetic field. The dynamics of electrons and photons is then described effectively by a two-dimensional Euclidean Schwinger model, where in a mechanism similar to the Higgs mechanism the photons acquire a finite mass  $m_\gamma \sim \frac{e}{L_B}$ , similar to the Schwinger's photon mass  $M_\gamma \sim g$ . Note that whereas in  $m_\gamma$ ,  $e$  is the dimensionless coupling constant of the  $3+1$  dimensional QED,  $g$  in  $M_\gamma$  is the dimensionful coupling constant of the two-dimensional QED. Moreover, in  $m_\gamma$ , the magnetic length  $L_B$  is defined by  $L_B \equiv |eB|^{-1/2}$ . Remarkable, however, is the fact that the effective two-dimensional field theory in the presence of strong magnetic field can be effectively described by a deformed noncommutative field theory in two spatial dimensions, where, for  $eB > 0$ , the commutation relation between the coordinates is given by  $[x_i, x_j] = \kappa_{ij}$  with  $\kappa_{ij} = L_B^2 (i\epsilon_{ij} - \delta_{ij})$ ,  $i, j = 1, 2$  [15]. This turns out to be similar to the corresponding relation in Euclidean noncommutative Schwinger model, with the noncommutativity parameter  $\theta \sim L_B^2$  playing the rôle of the squared of the magnetic length  $L_B$ . What concerns the noncommutative version of the Euclidean Schwinger model, we therefore expect that noncommutative photons acquire a finite mass, that depends in a non-trivial way on the dimensionful coupling constant  $g$  and the noncommutativity parameter  $\theta$ . It is the purpose of this paper to determine these noncommutative corrections to the commutative photon mass  $M_\gamma$ .

Another point, which makes this problem far from trivial, is the fact that noncommutative  $\text{QED}_2$  has similar properties to commutative  $\text{QCD}_2$ . This is because of the presence of noncommutative star-products in the gauge kinetic term of the Lagrangian density of the theory, including cubic and quartic couplings of noncommutative gauge fields, as in commutative  $\text{QCD}_2$ . Moreover, the equation of motion of noncommutative  $\text{QED}_2$  includes, similar to commutative  $\text{QCD}_2$ , a covariant derivative of the noncommutative field strength tensor and is therefore non-linear in the gauge field. Whereas in the commutative Schwinger model, the photon mass  $M_\gamma$  is determined perturbatively from the pole of photon propagator, and turns out to be one-loop exact [13], the mass spectrum of  $\text{QCD}_2$  is non-trivial and looking for it has a long history. The mass spectrum of pure commutative two-dimensional single flavor  $U(N_c)$  model was first determined by 't Hooft [16], where in the  $N_c \rightarrow \infty$  limit a nearly straight Regge trajectory of confined mesonic states was found. Recently, the spectrum of multi-flavor commutative  $SU(N_c)$  in two dimensions is determined in [17]. Here, the authors consider first a (commutative) gauged bosonized action equivalent to  $\text{QCD}_2$  action, that consists of a free Wess-

Zumino-Witten (WZW) part, a gauge kinetic part and an appropriate interaction part including the WZW and gauge fields. Using then the light-cone momentum operator, depending on the mesonic currents, they derive a 't Hooft-like mass eigenvalue equation  $P^2|\Phi\rangle = M^2|\Phi\rangle$ , where  $|\Phi\rangle$  is the wave function of “currentball” states of QCD<sub>2</sub> in a Hilbert space which is truncated to two-current states [17]. Solving this eigenvalue equation numerically, they arrive at the mass spectrum of QCD<sub>2</sub>, that, in particular, in the limit of large  $N_f \gg N_c$ , reduces to a single noninteracting meson with mass  $m = N_f M_\gamma$ , i.e.  $N_f$  copies of the commutative Schwinger mass  $M_\gamma = \frac{g}{\sqrt{\pi}}$ . In the present paper, we will closely follow the arguments of [17], and will determine the mass spectrum of noncommutative QED<sub>2</sub> nonperturbatively. We will arrive at the quite unexpected result that the mass-spectrum of noncommutative QED<sub>2</sub> does not receive any noncommutative correction and is therefore given by the spectrum of commutative QED<sub>2</sub>, consisting of a single free meson with mass  $M_\gamma = \frac{g}{\sqrt{\pi}}$ . We will show that this result is perturbatively exact in the noncommutativity parameter  $\theta$ , and the noncommutative QED<sub>2</sub> coupling constant  $g$ . It is, however, limited to the two current sector of the Hilbert space. This is in contrast with the results arising in [18]. Here, a noncommutative 1 + 1 dimensional bosonized Schwinger model is considered. Using a perturbative Seiberg-Witten map [2] up to first order in  $\theta$ , it is then mapped into an equivalent gauge invariant commutative model. It is shown that the resulting deformed theory consists of a massive boson, which is, in contrast to its noncommutative counterpart, no longer free. The same noncommutative bosonized Schwinger model, as in [18], is also considered in [19], where, in particular, the confinement-deconfinement phase transition observed in commutative Schwinger model is investigated. It is shown that though the fuzziness of space-time introduces new features in the confinement scenario, it does not affect the deconfining limit [19]. Schwinger model on fuzzy spheres is studied in [20]. Other exactly solvable noncommutative models are discussed recently in [21] (see also the references therein).

This paper is organized as follows: In Sec. II, after introducing the noncommutative QED<sub>2</sub> model, we will determine the one-loop correction to photon mass by determining the pole of the corresponding photon propagator at one-loop level. In this order, although additional diagrams relative to commutative QED<sub>2</sub> are to be taken into account and they can potentially contribute to the pole of the noncommutative photon propagator, the noncommutative photon mass, being the same as the commutative photon mass, receives no noncommutative corrections. To show that this result is one-loop exact, we will follow, in Sec. III, the method used in [17] and determine the mass spectrum of noncommutative QED<sub>2</sub> nonperturbatively. In Sec. III.A, following the method introduced in [22, 23] to bosonize commutative QCD<sub>2</sub>, we will first present the noncommutative Polyakov-Wiegmann fermionic effective action and the full gauged bosonized action of noncommutative QED<sub>2</sub>, that includes, in particular,

a noncommutative WZW part. The latter coincides with the WZW action previously determined in [24] by explicitly computing the fermion determinant of noncommutative QED<sub>2</sub>. In Sec. III.B, we will derive the energy-momentum tensor corresponding to the gauged bosonized action of noncommutative QED<sub>2</sub>, which will then be used in Sec. III.C to determine the mass spectrum of noncommutative QED<sub>2</sub> by solving the corresponding mass eigenvalue equation. Section IV is devoted to our concluding remarks.

## II. PHOTON MASS IN NONCOMMUTATIVE QED<sub>2</sub>: PERTURBATIVE APPROACH

### A. The model

The Lagrangian density of noncommutative field theories are given by their commutative counterpart with the commutative product of functions replaced by noncommutative star-product, defined by

$$f(x) \star g(x) \equiv \exp \left( \frac{i\theta_{\mu\nu}}{2} \frac{\partial}{\partial \xi_\mu} \frac{\partial}{\partial \zeta_\nu} \right) f(x + \xi)g(x + \zeta) \Big|_{\xi=\zeta=0}. \quad (\text{II.1})$$

Here,  $\theta_{\mu\nu}$  is an antisymmetric constant matrix reflecting the noncommutativity of space-time coordinates  $x_\mu$  and  $x_\nu$ ,  $[x_\mu, x_\nu]_\star \equiv i\theta_{\mu\nu}$ . In two-dimensional space-time coordinates  $\theta_{01}$  and  $\theta_{10} = -\theta_{01} = -\theta$  are given in terms of antisymmetric tensor of rank two  $\epsilon_{\mu\nu}$  as  $\theta_{\mu\nu} = \theta\epsilon_{\mu\nu}$ . In this paper, noncommutative QED<sub>2</sub> will be considered in two-dimensional Euclidean space, where  $x^0 = -ix^2$ . The Lagrangian density of two-dimensional noncommutative massless QED is then given by

$$\mathcal{L} = -i\bar{\psi} \star \gamma_\mu \partial^\mu \psi + g\bar{\psi} \star \gamma_\mu A^\mu \star \psi + \frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu) \star (\partial_\nu A^\nu) - \partial_\mu \bar{c} \star (\partial^\mu c - ie[A_\mu, c]_\star), \quad (\text{II.2})$$

with the Dirac  $\gamma$ -matrices  $\gamma_1 = \sigma_2$ , and  $\gamma_2 = -\sigma_1$ , satisfying  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$  and  $[\gamma_\mu, \gamma_\nu] = 2i\epsilon_{\mu\nu}\gamma_5$ , where  $\gamma_5 = -i\gamma_1\gamma_2 = \sigma_3$ . Here,  $\sigma_i, i = 1, 2, 3$  are Pauli matrices. In (II.2),  $\xi$  is the gauge fixing parameter and the field strength tensor  $F_{\mu\nu}$  is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]_\star, \quad (\text{II.3})$$

with  $[A_\mu, A_\nu]_\star = A_\mu \star A_\nu - A_\nu \star A_\mu$ . The Lagrangian density (II.2) is invariant under *global*  $U(1)$  transformation  $\delta_\alpha \psi = i\alpha\psi$  and  $\delta_\alpha \bar{\psi} = -i\alpha\bar{\psi}$ , which leads, according to the prescriptions in [25], to two different Noether currents

$$J_\mu(x) = \psi(x) \star \bar{\psi}(x) \gamma_\mu, \quad (\text{II.4})$$

$$j_\mu(x) = \bar{\psi}(x) \gamma_\mu \star \psi(x). \quad (\text{II.5})$$

Depending on their transformation properties under *local*  $U(1)$  transformation,  $\psi(x) \rightarrow e^{ig\alpha(x)} \star \psi(x)$ , the currents in (II.4) and (II.5) are denoted by covariant and invariant currents, respectively. They satisfy the classical continuity equations

$$D_\mu J^\mu(x) = 0, \quad \text{and} \quad \partial_\mu j^\mu(x) = 0, \quad (\text{II.6})$$

with  $D_\mu J^\mu = \partial_\mu J^\mu - ig[A_\mu, J^\mu]_\star$ . They arise from the equations of motion of massless fermionic fields

$$\partial_\mu \bar{\psi} \gamma^\mu = ig \bar{\psi} \gamma^\mu \star A_\mu, \quad \text{and} \quad \gamma^\mu \partial_\mu \psi = -ig A_\mu \star \gamma^\mu \psi. \quad (\text{II.7})$$

Similarly, there are two different axial vector currents

$$J_\mu^5(x) = \psi(x) \star \bar{\psi}(x) \gamma_\mu \gamma_5, \quad (\text{II.8})$$

$$j_\mu^5(x) = \bar{\psi}(x) \gamma_\mu \gamma_5 \star \psi(x), \quad (\text{II.9})$$

associated with the global  $U_A(1)$  axial invariance of the Lagrangian density (II.2) under  $\delta_\alpha \psi = i\alpha \gamma_5 \psi$  and  $\delta_\alpha \bar{\psi} = i\alpha \gamma_5 \bar{\psi}$ . They satisfy, similar to the vector currents (II.4) and (II.5), two different classical conservation laws

$$D^\mu J_\mu^5(x) = 0, \quad \text{and} \quad \partial^\mu j_\mu^5(x) = 0, \quad (\text{II.10})$$

that can be derived also using the relation  $\gamma^\mu \gamma^5 = -i\epsilon^{\mu\nu} \gamma_\nu$ , which is satisfied only in two dimensions, and the continuity relations (II.6). In Sec. III, we will use the covariant axial vector current (II.8), whose axial anomaly is given by [26]

$$D^\mu J_\mu^5 = -\frac{g}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}, \quad (\text{II.11})$$

to bosonize noncommutative QED<sub>2</sub>.<sup>1</sup>

## B. One-loop correction to the photon mass in noncommutative Schwinger model

It is the purpose of this paper to determine possible noncommutative corrections to the photon mass in two-dimensional noncommutative Euclidean space. In this section, we will determine the noncommutative photon mass perturbatively, by computing the pole of noncommutative photon propagator at one-loop level. In ordinary commutative Schwinger model, the photon mass is similarly determined perturbatively from the pole of the full photon propagator

$$\mathcal{D}^{\mu\nu}(p) = -\frac{1}{p^2[1 + \Pi(p^2)]} \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) - \xi \frac{p^\mu p^\nu}{p^4}, \quad (\text{II.12})$$

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<sup>1</sup> For the axial anomaly of the invariant axial vector current  $j_\mu^A$  in two dimensions see [27]. Nonplanar anomaly in  $d = 4$  dimensions was first determined in [25].

where,  $\Pi(p^2)$  is the scalar function appearing in the photon self-energy (the vacuum polarization tensor),  $\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu) \Pi(p^2)$ . At one-loop level,  $\Pi(p^2)$  is given by dimensionally regularized Feynman integral

$$\Pi(p^2) = -\frac{2g^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \, x(1-x) \frac{\Gamma(2 - \frac{d}{2})}{(-x(1-x)p^2)^{2-\frac{d}{2}}}, \quad (\text{II.13})$$

corresponding to the one-loop correction to the photon propagator from Fig. 1.

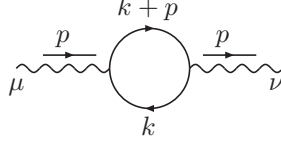


FIG. 1: Relevant one-loop photon self-energy diagram in commutative QED.

In  $d = 2$  dimensions, the Feynman integral (II.13) turns out to be finite and yields  $\Pi(p^2) = \frac{g^2}{\pi p^2}$  with a pole in the limit  $p^2 \rightarrow 0$ . Plugging  $\Pi(p^2)$  in (II.12), the photon propagator reads

$$\mathcal{D}^{\mu\nu}(p) = -\frac{1}{p^2 - \mu^2} \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) - \xi \frac{p^\mu p^\nu}{p^4}, \quad (\text{II.14})$$

with the pole  $\mu^2 = \frac{g^2}{\pi}$ , which can be interpreted as the commutative photon mass  $M_\gamma = \mu$ , and turns out to be one-loop exact [13]. Following the same method, we will now determine the photon mass in two-dimensional noncommutative Schwinger model at one-loop level. The general structure of photon self-energy in  $d$ -dimensions is studied previously in [28], without referring directly to the photon mass in two-dimensions. In the subsequent paragraph, we will present a brief review of these results, and postpone the details of the computations to App. A.

As it is argued in [28], in  $d$ -dimensions, the general structure of the photon propagator is given by inverting the 1PI two-point function and is given by

$$D_{\mu\nu} = -\frac{1}{p^2 + A} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \right) - \frac{1}{p^2 + A + B} \left( \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \right) - \xi \frac{p_\mu p_\nu}{p^4}. \quad (\text{II.15})$$

Here,  $\tilde{p}^\mu = \theta^{\mu\nu} p_\nu$ . Moreover,  $A$  and  $B$  are form factors arising from the vacuum polarization tensor  $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu} = A \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + B \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2}. \quad (\text{II.16})$$

Perturbatively, they receive contributions from Feynman integrals corresponding to the vacuum polarization tensor  $\Pi^{\mu\nu}$ . They are scalar functions in  $p^2$  and  $\tilde{p}$ . In Fig. 2, the relevant one-loop diagrams contributing to  $A$  and  $B$  in noncommutative QED are shown.

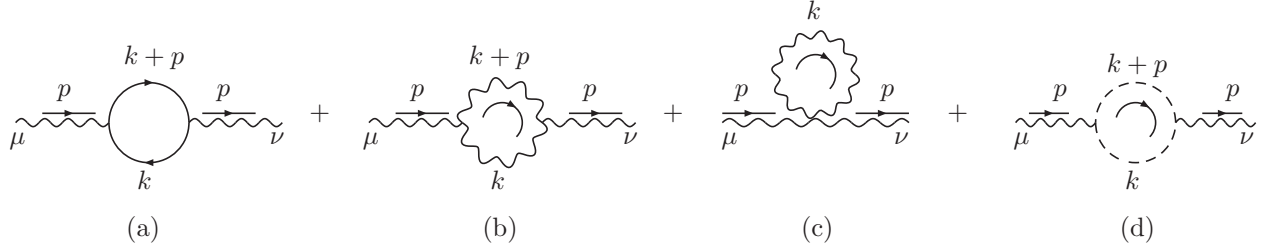


FIG. 2: Relevant one-loop photon self-energy diagrams in noncommutative QED.

Using the relation  $\theta^{\mu\nu} = \theta\epsilon^{\mu\nu}$ , which is valid only in  $d = 2$  dimensions, it is easy to show that  $\tilde{p}_\mu \tilde{p}_\nu = \theta^2(p^2\delta_{\mu\nu} - p_\mu p_\nu)$ . Plugging this relation in the vacuum polarization tensor (II.16) and the photon propagator  $D^{\mu\nu}$  from (II.15), we get

$$\Pi^{\mu\nu} = (A + B) \left( \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right), \quad (\text{II.17})$$

as well as

$$D_{\mu\nu} = -\frac{1}{p^2 + (A + B)} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - \xi \frac{p_\mu p_\nu}{p^4}. \quad (\text{II.18})$$

Identifying  $A + B$  in the denominator of (II.18) with  $A + B \equiv p^2 \Pi(p^2)$ , the noncommutative photon propagator (II.18) is comparable with (II.12) from commutative QED. Thus, according to our description at the beginning of this section, in order to determine the one-loop correction to the photon mass in noncommutative field theory in two dimensions, it is enough to look at  $p^2 \rightarrow 0$  behavior of  $A + B$ . Computing the corresponding Feynman integrals to the diagrams of Fig. 2, we arrive after a lengthy but straightforward calculation at (see App. A for more details)

$$\begin{aligned} A + B = & -\mu^2 - \frac{\mu^2}{4} \int_0^1 dx (x(1-x))^{-1} \left\{ 2(1+2x)(sK_1(s) - 1) \right. \\ & - (1-\xi) [6(2x^2 - 3x + 1)(sK_1(s) - 1) + (4x^2 - 6x + 1)(s^2K_2(s) - 2)] \\ & \left. - \frac{1}{4}(1-\xi)^2(5s^2K_2(s) - 2 - s^3K_3(s)) \right\}, \end{aligned} \quad (\text{II.19})$$

where  $\mu^2 \equiv \frac{g^2}{\pi}$ , and  $K_n(s)$  are the modified Bessel-function of  $n$ -th order in terms of  $s \equiv p^2\theta\sqrt{x(1-x)}$ . Keeping the noncommutativity parameter  $\theta$  finite and taking the IR limit  $p^2 \rightarrow 0$ , or equivalently  $s \rightarrow 0$ , we get

$$\lim_{p^2 \rightarrow 0} A + B = \lim_{p^2 \rightarrow 0} p^2 \Pi(p^2) = -\mu^2. \quad (\text{II.20})$$

This is in particular, because of the relation  $s^n K_n(s) = 2^{n-1} \Gamma[n] + \mathcal{O}(s^2)$ . Plugging (II.20) in (II.18), it turns out that in noncommutative Schwinger model the photon mass at one-loop level is still given by

$$M_\gamma \equiv \mu = \frac{g}{\sqrt{\pi}}, \quad (\text{II.21})$$



and does not receive any correction proportional to the noncommutative parameter  $\theta$ . The question of whether this result is, as in the commutative case, exact in the orders of  $\theta$  and  $g$  can not be answered within the framework of perturbation theory. In the next section, we will bosonize the theory; limiting ourselves to the two-current sector of the Hilbert space and solving the corresponding mass eigenvalue equation, we will show that the above result is indeed exact in noncommutative parameter  $\theta$  and coupling constant  $g$ .

### III. PHOTON MASS IN NONCOMMUTATIVE QED<sub>2</sub>: NONPERTURBATIVE APPROACH

In the first part of this section, we will briefly review the method leading to an appropriate noncommutative bosonized action, which is equivalent to the fermionic noncommutative action. For this purpose, we follow closely the method introduced in [22] and [23]. In the second part, the corresponding energy-momentum tensor for the noncommutative bosonized gauged action, will be determined. Finally, we will solve the mass eigenvalue equation of the theory and determine the noncommutative photon mass nonperturbatively.

#### A. Polyakov-Wiegmann functional and the gauged bosonized action of noncommutative Schwinger model

In this section, we will derive the noncommutative Polyakov-Wiegmann functional.<sup>2</sup> Using the corresponding properties of this functional under local vector and axial vector gauge transformation, the gauged WZW action of noncommutative Schwinger model will be derived, that, after inclusion of the gauge kinetic part of noncommutative photons will eventually lead to the bosonized action of noncommutative Schwinger model. To do this, we start, as in commutative QCD<sub>2</sub>, by choosing the gauge field  $A_{\pm}$  in the light-cone coordinates<sup>3</sup> in terms of two fields  $U$  and  $V$  as

$$A_+ = \frac{i}{g} U^{-1} \star \partial_+ U, \quad \text{and} \quad A_- = \frac{i}{g} V \star \partial_+ V^{-1}, \quad (\text{III.1})$$

where,  $U$  and  $V$  satisfy  $U \star U^{-1} = 1$  and  $V \star V^{-1} = 1$ . Moreover, under noncommutative  $U(1)$  gauge transformation,  $U$  and  $V$  transform as

$$U \rightarrow U \star g_v^{-1}, \quad \text{and} \quad V \rightarrow g_v \star V, \quad (\text{III.2})$$

<sup>2</sup> See [22, 23, 29] for the commutative counterpart of this functional in QCD<sub>2</sub>.

<sup>3</sup> The light-cone coordinates are defined by  $x_{\pm} = x_1 \pm ix_2$ . Similarly,  $A_{\pm} = A_1 \pm iA_2$ .

which is equivalent with the familiar  $U(1)$  gauge transformation of the gauge fields

$$A_{\pm} \rightarrow g_v \star A_{\pm} \star g_v^{-1} + \frac{i}{g} g_v \star \partial_{\pm} g_v^{-1}, \quad (\text{III.3})$$

with  $A_{\pm}$  from (III.1), and  $g_v \star g_v^{-1} = 1$ . To determine  $J_{\pm}$  in terms of  $U$  and  $V$ , we use first the gauge invariance of the fermionic determinant and choose, without loss of generality,  $V = 1$  or equivalently  $A_- = 0$ . Then, solving the continuity equation (II.6) and the anomaly equation (II.11) in the light-cone coordinates and in  $A_- = 0$  gauge,

$$\begin{aligned} \partial_+ J_- + \partial_- J_+ - ig[A_+, J_-]_{\star} &= 0, \\ \partial_- J_+ - \partial_+ J_- + ig[A_+, J_-]_{\star} &= \frac{g}{\pi} \partial_- A_+, \end{aligned} \quad (\text{III.4})$$

we arrive at

$$J_{\pm} = \pm \frac{i}{2\pi} U^{-1} \star \partial_{\pm} U. \quad (\text{III.5})$$

The effective action  $W[A]$  is then obtained using  $J_- = \frac{2}{g} \frac{\delta W[A]}{\delta A_+}$ ,

$$\delta W[A] = \frac{g}{2} \int d^2x J_-(x) \star \delta A_+(x). \quad (\text{III.6})$$

Varying  $A_+$  from (III.1), we get

$$-ig\delta A_+ = D_+(U^{-1} \star \delta U), \quad \text{where} \quad D_+ f \equiv \partial_+ f + [U^{-1} \star \partial_+ U, f]_{\star}. \quad (\text{III.7})$$

Plugging  $\delta A_+$  from (III.7) together with  $J_-$  from (III.5) in (III.6), and using, after integrating (III.6) by part, the identity  $D_-(U^{-1} \star \partial_+ U) = \partial_+(U^{-1} \star \partial_- U)$ ,  $\delta W$  can be rewritten as

$$\delta W = -\frac{1}{4\pi} \int d^2x (U^{-1} \star \delta U) \star \partial_-(U^{-1} \star \partial_+ U). \quad (\text{III.8})$$

This equation may then be integrated in terms of the WZW effective action,  $\Gamma[U] = -W[A]$ , which consists of two parts,  $\Gamma[U] = S_{\text{P}\sigma\text{M}}[U] + S_{\text{WZ}}[U]$ . Here,  $S_{\text{P}\sigma\text{M}}[U]$  is the action of the principal  $\sigma$ -model, which is given by

$$S_{\text{P}\sigma\text{M}}[U] = \frac{1}{8\pi} \int d^2x (\partial_{\mu} U^{-1}) \star (\partial^{\mu} U) = -\frac{1}{8\pi} \int d^2x (U \star \partial_{\mu} U^{-1}) \star (U \star \partial^{\mu} U^{-1}), \quad (\text{III.9})$$

and  $S_{\text{WZ}}[U]$  is the Wess-Zumino term, that reads

$$S_{\text{WZ}}[U] = -\frac{i}{4\pi} \int_0^1 dr \int d^2x \epsilon^{\mu\nu} U_r^{-1} \star \dot{U}_r \star U_r^{-1} \star \partial_{\mu} U_r \star U_r^{-1} \star \partial_{\nu} U_r. \quad (\text{III.10})$$

In (III.10),  $U_r$  is the extension of  $U$  and satisfies the boundary conditions  $U_r(x)|_{r=1} = U$ , and  $U_r(x)|_{r=0} = 1$ . Moreover,  $\dot{U}_r \equiv \partial_r U_r$ . To obtain the noncommutative Polyakov-Wiegmann functional, the effective action  $\Gamma[U]$  is to be redefined in terms of gauge invariant combination  $\Sigma \equiv U \star V$ .

We arrive first at  $\Gamma[\Sigma] = S_{\text{P}\sigma\text{M}}[\Sigma] + S_{\text{WZ}}[\Sigma]$ , with the principal  $\sigma$ -model part from (III.9),

$$S_{\text{P}\sigma\text{M}}[\Sigma] = S_{\text{P}\sigma\text{M}}[U] + S_{\text{P}\sigma\text{M}}[V] + \frac{1}{4\pi} \int d^2x (U^{-1} \star \partial_{\mu} U) \star (V \star \partial^{\mu} V^{-1}), \quad (\text{III.11})$$

and the Wess-Zumino part from (III.10)

$$S_{\text{WZ}}[\Sigma] = S_{\text{WZ}}[U] + S_{\text{WZ}}[V] - \frac{i}{4\pi} \int_0^1 dr \int d^2x \epsilon^{\mu\nu} \mathcal{W}_{\mu\nu}, \quad (\text{III.12})$$

with

$$\mathcal{W}_{\mu\nu} = \frac{d}{dr} (U_r^{-1} \star \partial_\mu U_r \star V_r \star \partial_\nu V_r^{-1}) - \partial_\mu (V_r \star \partial_\nu V_r^{-1} \star U_r^{-1} \star \dot{U}_r) - \partial_\nu (U_r^{-1} \star \partial_\mu U_r \star V_r \star \dot{V}_r^{-1}). \quad (\text{III.13})$$

Neglecting then the last two terms in (III.13), that yield two vanishing surface integrals after integrating over the coordinates in (III.12), and using the boundary conditions for  $U_r$  and  $V_r$ ,<sup>4</sup> we arrive at the noncommutative Polyakov-Wiegmann functional for  $\Gamma[\Sigma]$ ,

$$\Gamma[\Sigma] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi} \int d^2x (\delta^{\mu\nu} - i\epsilon^{\mu\nu}) (U^{-1} \star \partial_\mu U) \star (V \star \partial_\nu V^{-1}), \quad (\text{III.14})$$

or equivalently in the light-cone coordinates, at

$$\Gamma[\Sigma] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi} \int d^2x (U^{-1} \star \partial_+ U) \star (V \star \partial_- V^{-1}). \quad (\text{III.15})$$

Note that to derive the factorized forms (III.11) and (III.12) of  $S_{\text{P}\sigma\text{M}}[\Sigma]$  as well as  $S_{\text{WZ}}[\Sigma]$ , extensive use is made of the trace property of the star-product of two functions under a two-dimensional integral  $\int d^2x f \star g = \int d^2x g \star f$ , and the relations  $\partial_\mu U \star U^{-1} = -U \star \partial_\mu U^{-1}$ , as well as  $\partial_\mu V \star V^{-1} = -V \star \partial_\mu V^{-1}$ , arising from  $U \star U^{-1} = 1$  and  $V \star V^{-1} = 1$ . Although the Polyakov-Wiegmann functional (III.14) is invariant under gauge transformation (III.2), it is not invariant under the  $U_A(1)$  axial gauge transformation

$$U \rightarrow U \star g_a^{-1}, \quad \text{and} \quad V \rightarrow g_a^{-1} \star V, \quad (\text{III.16})$$

or equivalently

$$\begin{aligned} A_+ &\rightarrow g_a \star A_+ \star g_a^{-1} + \frac{i}{g} g_a \star \partial_+ g_a^{-1}, \\ A_- &\rightarrow g_a^{-1} \star A_- \star g_a + \frac{i}{g} g_a^{-1} \star \partial_- g_a, \end{aligned} \quad (\text{III.17})$$

with  $g_a \star g_a^{-1} = 1$ . This non-invariance can be expressed in an equivalent gauged bosonic action for the fermions, defined by

$$S_F[A, \omega] \equiv \Gamma[\Sigma; \omega] - \Gamma[\Sigma], \quad (\text{III.18})$$

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<sup>4</sup> Here, it is assumed that  $V_r$  satisfies the same boundary conditions as  $U_r$ .

Here, the bosonized action  $S_F[A, \omega]$  remains invariant under local gauge transformation (III.2) of  $U$  and  $V$  fields and the vector gauge transformation of the WZW field  $\omega$ , i.e.  $\omega \rightarrow g_v \star \omega \star g_v^{-1}$  [22]. The explicit form of  $S[A, \omega]$  can be found by successive use of the Polyakov-Wiegmann functional in the light-cone coordinates (III.15), and is given by

$$S_F[A, \omega] = \Gamma[\omega] + \frac{1}{4\pi} \int d^2x \mathcal{W}[A, \omega], \quad (\text{III.19})$$

with  $\mathcal{W}[A, \omega]$  defined by

$$\mathcal{W}[A, \omega] \equiv g^2 A_+ \star A_- - g^2 A_+ \star \omega \star A_- \star \omega^{-1} - ig A_+ \star \omega \star \partial_- \omega^{-1} - ig A_- \star \omega^{-1} \star \partial_+ \omega, \quad (\text{III.20})$$

where the definitions of  $A_{\pm}$  in terms of  $U$  and  $V$  from (III.1) are used. Using  $\mathcal{J}_{\pm} \equiv \frac{2}{g} \frac{\delta S_F[A, \omega]}{\delta A_{\mp}}$ , with  $S_F[A, \omega]$  from (III.19)-(III.20), the corresponding currents are given by

$$\begin{aligned} \mathcal{J}_+ &= \frac{1}{2\pi} (g A_+ - g \omega^{-1} \star A_+ \star \omega - i \omega^{-1} \star \partial_+ \omega), \\ \mathcal{J}_- &= \frac{1}{2\pi} (g A_- - g \omega \star A_- \star \omega^{-1} - i \omega \star \partial_- \omega^{-1}). \end{aligned} \quad (\text{III.21})$$

Note that in light-cone gauge  $A_- = 0$ ,  $\mathcal{J}_-$  from (III.21) is reduced to

$$\mathcal{J}_- = J_-[\omega] = -\frac{i}{2\pi} \omega \star \partial_- \omega^{-1}$$

which is of the same form as the currents  $J_-$  from (III.5) of the free theory, rewritten in terms of  $\omega$ . Same result arises also in commutative QCD<sub>2</sub> in the light-cone gauge [30].<sup>5</sup> Moreover, it can be shown that since  $J_{\pm}$  satisfy the equations

$$\partial_{\mp} J_{\pm} = 0, \quad (\text{III.22})$$

arising from the variation of  $\Gamma[\omega]$ ,  $\mathcal{J}_-$  in light-cone gauge is, as in QCD<sub>2</sub>, only a function of  $x^-$ . Later, we will use this fact to show that the spectrum of noncommutative QED<sub>2</sub> reduces to the spectrum of commutative Schwinger model. Adding, at this stage, the gauge kinetic action  $\sim \int d^2x F^{\mu\nu} F_{\mu\nu}$  to (III.19), we arrive at the full gauged bosonized action of noncommutative QED<sub>2</sub>

$$\begin{aligned} S_b[A, \omega] &= \Gamma[\omega] + \frac{1}{4} \int d^2x F_{\mu\nu} \star F^{\mu\nu} \\ &+ \frac{1}{4\pi} \int d^2x \left\{ g^2 A_{\mu} \star A^{\mu} - g^2 \kappa_-^{\mu\nu} A_{\mu} \star \omega \star A_{\nu} \star \omega^{-1} - ig \kappa_-^{\mu\nu} A_{\mu} \star \omega \star \partial_{\nu} \omega^{-1} - ig \kappa_+^{\mu\nu} A_{\mu} \star \omega^{-1} \star \partial_{\nu} \omega \right\}, \end{aligned} \quad (\text{III.23})$$

<sup>5</sup> Similarly one can show that since in the light-cone gauge  $S_F[A, \omega]$  from (III.19) reduces to  $S_F[A, \omega] = \Gamma[\omega] - \frac{ig}{4\pi} \int d^2x A_+ \star \omega \star \partial_- \omega^{-1}$ , the only current coupled to  $A_+$  is  $\mathcal{J}_- = \frac{2}{g} \frac{\delta S_F}{\delta A_+} = -\frac{i}{2\pi} \omega \star \partial_- \omega = J_-[\omega]$ .

where  $\kappa_{\pm}^{\mu\nu} \equiv \delta^{\mu\nu} \pm i\epsilon^{\mu\nu}$ . In the light-cone gauge  $A_- = 0$ , the bosonized action (III.23) is further reduced to

$$S_b[\omega] = \Gamma[\omega] - \frac{g^2}{2} \int d^2x \, J_- \star \frac{1}{\partial_-^2} J_-. \quad (\text{III.24})$$

Here,  $J_- = -\frac{i}{2\pi} \omega \star \partial_- \omega^{-1}$ . To arrive at (III.24), we have replaced  $F_{\mu\nu} F^{\mu\nu}$  in (III.23) with  $F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\partial_- A_+)^2$  and used the equation of motion of the gauge fields  $D_\mu F^{\mu\nu} = -gJ^\nu$ , that in the light-cone gauge reduces to  $\partial_-^2 A_+ = 2gJ_-$ . In the next section, the energy-momentum tensor corresponding to (III.23) will be derived.

## B. Energy-momentum tensor

To determine the mass spectrum of noncommutative Schwinger model, we have to solve the eigenvalue equation  $P^2|\Phi\rangle = M^2|\Phi\rangle$ , that in light-cone space reduces to  $P^+P^-|\Phi\rangle = M^2|\Phi\rangle$ , with  $P^\pm \sim \int dx^- T^{+\pm}$ . Here,  $T^{+\pm}$  are the components of the total energy-momentum tensor  $T^{\mu\nu}$  corresponding to the full bosonized action (III.23) in the light-cone coordinates. In this section, we use the general definition of  $T^{\mu\nu}$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \Big|_{g_{\mu\nu}=\delta_{\mu\nu}}, \quad (\text{III.25})$$

to determine the total energy momentum tensor

$$T^{\mu\nu} = T_{\text{P}\sigma\text{M}}^{\mu\nu} + T_{\text{WZ}}^{\mu\nu} + T_{\text{gauge}}^{\mu\nu} + T_{\text{W}}^{\mu\nu}. \quad (\text{III.26})$$

Plugging first the action of the principal  $\sigma$ -model  $S_{\text{P}\sigma\text{M}}[\omega]$  from (III.9) in (III.25), the corresponding energy momentum-tensor  $T_{\text{P}\sigma\text{M}}^{\mu\nu}$  in a Sugawara form [31] reads

$$T_{\text{P}\sigma\text{M}}^{\mu\nu} = \frac{\pi}{2} \left( J^\mu(x) \star J^\nu(x) + J^\nu(x) \star J^\mu(x) - \delta^{\mu\nu} J_\lambda(x) \star J^\lambda(x) \right). \quad (\text{III.27})$$

In the light-cone gauge  $A_- = 0$ , where  $J^+ = J_- = -\frac{i}{2\pi} \omega \star \partial_- \omega^{-1}$ , the components of  $T_{\text{P}\sigma\text{M}}^{\mu\nu}$  are given by

$$T_{\text{P}\sigma\text{M}}^{++} = \pi J^+ \star J^+, \quad \text{and} \quad T_{\text{P}\sigma\text{M}}^{+-} = T_{\text{P}\sigma\text{M}}^{-+} = T_{\text{P}\sigma\text{M}}^{--} = 0. \quad (\text{III.28})$$

Similarly, the energy-momentum tensor  $T_{\text{WZ}}^{\mu\nu}$ , corresponding to the Wess-Zumino part of the effective action  $S_{\text{WZ}}[\omega]$  from (III.10) is determined using (III.25) and reads

$$T_{\text{WZ}}^{\mu\nu} = -\frac{i}{4\pi} \int_0^1 dr \int d^2x \left( -\delta^{\mu\nu} \epsilon^{\rho\sigma} \omega_r^{-1} \star \dot{\omega}_r \star \omega_r^{-1} \star \partial_\rho \omega_r \star \omega_r^{-1} \star \partial_\sigma \omega_r \right. \\ \left. + \epsilon^{\mu\sigma} \omega_r^{-1} \star \dot{\omega}_r \star \omega_r^{-1} \star \partial^\nu \omega_r \star \omega_r^{-1} \star \partial_\sigma \omega_r + \epsilon^{\sigma\mu} \omega_r^{-1} \star \dot{\omega}_r \star \omega_r^{-1} \star \partial_\sigma \omega_r \star \omega_r^{-1} \star \partial^\nu \omega_r \right). \quad (\text{III.29})$$

As it turns out, in the light-cone coordinates, all the components of the WZ part of the energy-momentum tensor vanish, in particular  $T_{\text{WZ}}^{++} = T_{\text{WZ}}^{+-} = 0$ . As for the noncommutative energy-momentum tensor corresponding to the gauge kinetic action  $\sim \int d^2x F_{\mu\nu}^2$  from (III.23), it is previously determined in [32], using the (III.25). It is given by

$$T_{\text{gauge}}^{\mu\nu} = \frac{1}{2} \left( F^{\nu\alpha} \star F_{\alpha}^{\mu} + F^{\mu\alpha} \star F_{\alpha}^{\nu} - \frac{1}{2} \delta^{\mu\nu} F^{\alpha\beta} \star F_{\alpha\beta} \right), \quad (\text{III.30})$$

and reduces in the light-cone gauge to

$$T_{\text{gauge}}^{++} = T_{\text{gauge}}^{--} = 0, \quad T_{\text{gauge}}^{+-} = T_{\text{gauge}}^{-+} = \frac{1}{8} (\partial^+ A^-) \star (\partial^+ A^-). \quad (\text{III.31})$$

Finally, the corresponding energy-momentum tensor to  $\mathcal{W}$  from (III.20) [or in  $\mu, \nu$  coordinates from the last term of (III.23)] is given by

$$\begin{aligned} T_{\mathcal{W}}^{\mu\nu} = \frac{1}{4\pi} \bigg\{ & \delta^{\mu\nu} \mathcal{W}(x) - g^2 (A^{\mu} \star A^{\nu} + A^{\nu} \star A^{\mu}) + g^2 (A^{\mu} \star \omega \star A^{\nu} \star \omega^{-1} + A^{\nu} \star \omega \star A^{\mu} \star \omega^{-1}) \\ & - i g^2 (\epsilon^{\mu\sigma} A^{\nu} \star \omega \star A_{\sigma} \star \omega^{-1} + \epsilon^{\sigma\mu} A_{\sigma} \star \omega \star A^{\nu} \star \omega^{-1}) + i g (A^{\mu} \star \omega \star \partial^{\nu} \omega^{-1} + A^{\nu} \star \omega \star \partial^{\mu} \omega^{-1}) \\ & + g (\epsilon^{\mu\sigma} A^{\nu} \star \omega \star \partial_{\sigma} \omega^{-1} + \epsilon^{\sigma\mu} A_{\sigma} \star \omega \star \partial^{\nu} \omega^{-1}) + i g (A^{\mu} \star \omega^{-1} \star \partial^{\nu} \omega + A^{\nu} \star \omega^{-1} \star \partial^{\mu} \omega) \\ & - g (\epsilon^{\mu\sigma} A^{\nu} \star \omega^{-1} \star \partial_{\sigma} \omega + \epsilon^{\sigma\mu} A_{\sigma} \star \omega^{-1} \star \partial^{\nu} \omega) \bigg\}. \end{aligned} \quad (\text{III.32})$$

It can be checked that in the light-cone gauge all the components of  $T_{\mathcal{W}}^{\mu\nu}$  vanish, in particular  $T_{\mathcal{W}}^{++} = T_{\mathcal{W}}^{+-} = 0$ . Adding all the contributions from (III.27), (III.29), (III.30) and (III.32) to the relevant component of the total energy-momentum tensor (III.25), we arrive at

$$T^{++} = T_{\text{PM}}^{++} = \bar{J} \star \bar{J}, \quad \text{and} \quad T^{+-} = T_{\text{gauge}}^{+-} = \frac{1}{8} (\partial^+ A^-) \star (\partial^+ A^-), \quad (\text{III.33})$$

where the identity  $J^+ = J_-$  and the redefinition  $\bar{J} \equiv \sqrt{\pi} J_-$  is used. Using (III.33),  $P^{\pm}$  are defined by

$$\begin{aligned} P^+ &= \frac{1}{2} \int dx^- T^{++} = \frac{1}{2} \int dx^- \bar{J}(x^-) \star \bar{J}(x^-) = \frac{1}{2} \int dx^- \bar{J}(x^-) \bar{J}(x^-), \\ P^- &= \frac{1}{2} \int dx^- T^{+-} = -\frac{g^2}{4\pi} \int dx^- \bar{J}(x^-) \star \frac{1}{\partial_-^2} \bar{J}(x^-) = -\frac{g^2}{4\pi} \int dx^- \bar{J}(x^-) \frac{1}{\partial_-^2} \bar{J}(x^-). \end{aligned} \quad (\text{III.34})$$

To arrive at  $P^-$ , we have used  $T^{+-}$  from (III.33), performed a partial integration, and used the equation of motion in the light-cone gauge,  $\partial_-^2 A_+ = 2g J_-$ . To remove the star-product in the integrand of  $P^{\pm}$ , we have used the property

$$\int d^d x f(x) \star g(x) = \int d^d x f(x) g(x),$$

of noncommutative star-product, and (III.22) stating that in the light-cone gauge  $J_-$  is only a function of  $x^-$ . Thus, it turns out that although the noncommutative nature of the theory is fully incorporated in the current  $\bar{J} \sim \omega \star \partial_- \omega^{-1}$ , the form of  $P^{\pm}$  is exactly the same as in commutative QED<sub>2</sub>. We will show in the subsequent section that this will be one of the crucial point to show that the mass spectrum of noncommutative QED<sub>2</sub> is exactly the same as in commutative Schwinger model.

### C. Mass spectrum of noncommutative QED<sub>2</sub>

As we have noted at the beginning of the previous section, to determine the mass spectrum of non-commutative Schwinger model, the mass eigenvalue equation  $P^2|\Phi\rangle = M^2|\Phi\rangle$  is to be solved. In the light-cone coordinates, this equation reduces to

$$P^+P^-|\Phi\rangle = M^2|\Phi\rangle, \quad (\text{III.35})$$

with  $P^\pm$  given in (III.34), which is derived in light-cone gauge  $A_- = 0$ . As for the eigenfunction  $|\Phi\rangle$ , we restrict ourselves, as in [17] for QCD<sub>2</sub>, to the two-current sector of the Hilbert space

$$|\Phi\rangle = \int_0^1 dk \Phi(k) \tilde{J}(-k) \tilde{J}(k-1) |0\rangle, \quad (\text{III.36})$$

where  $\tilde{J}(-k)$  and  $\tilde{J}(k-1)$  play the role of creation operators. In (III.36),  $\tilde{J}(k)$  is the Fourier transformation of  $\bar{J}(x^-)$ , defined by  $\tilde{J}(k^+) = \int \frac{dx^-}{\sqrt{2\pi}} e^{-ik^+x^-} \bar{J}(x^-)$ .<sup>6</sup> Using the Fourier transform of the current in momentum space,  $\tilde{J}(k)$ , the momenta  $P^\pm$  from (III.34) are given by

$$\begin{aligned} P^+ &= \int_0^\infty dp \tilde{J}(-k) \tilde{J}(k), \\ P^- &= \frac{g^2}{2\pi} \int_0^\infty dp \frac{1}{p^2} \tilde{J}(-p) \tilde{J}(p). \end{aligned} \quad (\text{III.37})$$

To solve (III.35), with  $P^\pm$  from (III.37) and  $|\Phi\rangle$  from (III.36), we use first the algebra of currents in Fourier-space [33]

$$[\tilde{J}(p), \tilde{J}(q)] = p \delta(p+q). \quad (\text{III.38})$$

Then, using  $P^\pm|0\rangle = 0$ , we arrive first at the relations

$$[P^+, \tilde{J}(-k)] = 2k\tilde{J}(-k), \quad \text{and} \quad [P^-, \tilde{J}(-k)] = \frac{g^2}{2\pi k} \tilde{J}(-k), \quad (\text{III.39})$$

that lead to

$$P^+|\Phi\rangle = 2 \int_0^1 dk \Phi(k) \tilde{J}(-k) \tilde{J}(k-1) |0\rangle = 2|\Phi\rangle, \quad (\text{III.40})$$

$$P^-|\Phi\rangle = \frac{g^2}{2\pi} \int_0^1 dk \Phi(k) \left( \frac{1}{k} + \frac{1}{1+k} \right) \tilde{J}(-k) \tilde{J}(k-1) |0\rangle. \quad (\text{III.41})$$

Here  $P^\pm$  from (III.37) are used. Multiplying (III.40) with  $M^2 (P^+)^{-1}$  and using the eigenvalue equation (III.35), we arrive at

$$P^-|\Phi\rangle = \frac{M^2}{2} |\Phi\rangle. \quad (\text{III.42})$$

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<sup>6</sup> To keep the notations as simple as possible, we have replaced  $k^+$  by  $k$  in (III.36).

Plugging now (III.41) and (III.36) in (III.42), we get

$$M^2 \Phi(k) = \frac{g^2}{\pi} \left( \frac{1}{k} + \frac{1}{1-k} \right) \Phi(k), \quad (\text{III.43})$$

that leads to the mass spectrum of the theory

$$M^2 = \frac{g^2}{\pi} \left( \frac{1}{k} + \frac{1}{1-k} \right). \quad (\text{III.44})$$

provided  $\Phi(k) \neq 0$ . This result is comparable with the mass spectrum of QCD<sub>2</sub> in  $N_f \gg N_c$  (see equation (35) in [17]). As in that case, (III.44) describes a continuum of states with masses above  $2M_\gamma$  in a theory whose spectrum reduces to a single non-interacting meson with mass  $M_\gamma = \frac{g}{\sqrt{\pi}}$  [17]. This supports our claim from the previous section, that in noncommutative QED<sub>2</sub>, photon's mass receives no  $\theta$ -dependent correction.

#### IV. CONCLUDING REMARKS

The fact that noncommutative QED<sub>2</sub> is, similar to commutative QED<sub>2</sub>, an exactly solvable model is not *a priori* clear. In this paper, we consider only QED in two-dimensional Euclidean space, with no possibility of going to Minkowski space. We have shown that the photon mass in noncommutative QED<sub>2</sub> does not receive any correction from the noncommutativity parameter  $\theta$ , and that the spectrum of noncommutative QED<sub>2</sub> reduces to a single noninteracting meson with mass  $M_\gamma = \frac{g}{\sqrt{\pi}}$ , as in commutative Schwinger model. To do this, in the first part of the paper, the pole of the noncommutative photon propagator is perturbatively computed in one-loop order. Although in noncommutative QED<sub>2</sub>, in contrast to commutative QED<sub>2</sub>, three additional Feynman integrals are to be considered, it is shown that the pole of the noncommutative photon propagator remains unchanged at this one-loop level. This result turns out to be (perturbatively) exact in the noncommutative parameter  $\theta$  and the coupling constant  $g$ , as in commutative QED<sub>2</sub>. It is, however, limited to the two-current sector of the Hilbert space. This is shown in the second part of the paper, by solving the mass eigenvalue equation of noncommutative QED<sub>2</sub> in the framework of an equivalent bosonized gauge theory in two-dimensional Euclidean space. Following closely the method used in [17] to determine the mass spectrum of QCD<sub>2</sub>, we have first derived the gauged bosonized effective action corresponding to noncommutative QED<sub>2</sub>, and its corresponding energy-momentum tensor in terms of noncommutative currents. Solving then the corresponding mass eigenvalue equation, we have determined the mass spectrum of noncommutative QED<sub>2</sub>, which turns out to be exactly the same as its commutative counterpart.

As we have argued in Sec. III, there are two fundamental reasons for this result. First, as we have seen in (III.34), in the light-cone gauge, the noncommutative momenta of the bosonized theory,  $P^\pm$ ,



turn out to have the same expressions in terms of  $\bar{J}$ , as their commutative counterparts. Using these  $P^\pm$ , we are therefore left with the same mass eigenvalue equation as in commutative QED<sub>2</sub>. Note that although the noncommutativity is fully incorporated in  $\bar{J} \sim \omega \star \partial_- \omega^{-1}$  in terms of the star-product, in the light-cone gauge, they depend only on one of the coordinates  $x^-$  [see (III.22)]. This is the main reason why under the integral over  $x^-$  the star-product appearing in the integrand of (III.34) can be removed between two  $\bar{J}$ 's, leaving us with the same  $P^\pm$  as in commutative Schwinger model. Being independent of  $x^+$ , the Fourier transform of  $\bar{J}$  depends only on  $k^+$ . This was our motivation behind the particular choice of (III.36) as the eigenstate  $|\Phi\rangle$ , which is restricted to two-current sector of the Hilbert space as in [17]. As for the second reason, it lies in the fact that the corresponding current algebra (III.38) of noncommutative QED<sub>2</sub>, being independent of the noncommutativity parameter  $\theta$ , is the same as in commutative QED<sub>2</sub>. This noncommutative current algebra was particularly used in Sec. III.C to determine certain commutation relations of  $P^\pm$  with the currents appearing in  $|\Phi\rangle$  [see (III.40) and (III.41)]. The latter are then used to determine the mass spectrum (III.44).

It would be intriguing to study the effect of nonplanar diagrams on noncommutative photon mass. This can be realized by bosonizing noncommutative QED<sub>2</sub> with *invariant* currents  $j_\mu$  and  $j_\mu^5$  from (II.5) and (II.9), in contrast to what is done in this paper. What changes is, in particular, the axial anomaly equation (II.11), corresponding to the covariant current  $J_\mu^5$ . As it is shown in [25, 27] the axial anomaly for invariant current  $j_\mu^5$  receives contributions from *nonplanar* diagrams. Together with the continuity equation, the anomaly equation (II.11) is particularly used in this paper to determine the exact form of the currents  $J_\pm$  in terms of the auxiliary field  $U$  in the light-cone gauge  $V = 1$  [see (III.5)]. Because of the form of nonplanar anomaly [25, 27], requiring an additional IR regularization [33], the study of the effects of nonplanar Feynman integrals on noncommutative photon mass is a non-trivial task and will be postponed to future publications.

## V. ACKNOWLEDGMENTS

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## Appendix A: Vacuum polarization tensor of noncommutative Schwinger model at one-loop level

In this appendix, we will determine the vacuum polarization tensor of noncommutative QED<sub>2</sub>. This is previously done in [28] for general  $d$ -dimensional case. All propagator and vertices can therefore be

read in [28].<sup>7</sup> The Feynman diagrams corresponding to the vacuum polarization tensor are presented in Fig. 2(a)-2(d). The corresponding Feynman integrals are given by

$$\begin{aligned}
\Pi_{(a)}^{\mu\nu}(p) &= -g^2 \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta)}{k^2(p+k)^2} k_\alpha(p+k)_\beta, \\
\Pi_{(b)}^{\mu\nu}(p) &= \frac{g^2 N_g}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} (A^{\mu\nu} + (1-\xi)B^{\mu\nu} + (1-\xi)^2 C^{\mu\nu}), \\
\Pi_{(c)}^{\mu\nu}(p) &= \frac{g^2 N_t}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} (k+p)^2 \left( (d-1)\delta^{\mu\nu} - (1-\xi)(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}) \right), \\
\Pi_{(d)}^{\mu\nu}(p) &= \frac{g^2 N_{gh}}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} k^\mu(p+k)^\nu,
\end{aligned} \tag{A.1}$$

where  $N_{gh} = 2, N_t = 2, N_g = 1, p \wedge k \equiv \theta^{\mu\nu} p_\mu k_\nu$ , and

$$\begin{aligned}
A^{\mu\nu} &= -(5p^2 + 2k^2 + 2k \cdot p)\delta^{\mu\nu} + (6-4d)k^\mu k^\nu + (6-d)p^\mu p^\nu + (3-2d)(k^\mu p^\nu + k^\nu p^\mu), \\
B^{\mu\nu} &= \left\{ \frac{1}{k^2} [(k^2 + 2k \cdot p)^2 \delta^{\mu\nu} - (k^2 + 2k \cdot p - p^2) k^\mu k^\nu - (k^2 + 3k \cdot p)(k^\mu p^\nu + k^\nu p^\mu) + p^\mu p^\nu k^2] \right\} \\
&\quad + \{k \rightarrow p+k, p \rightarrow -p\}, \\
C^{\mu\nu} &= -\frac{[p^2 k^\mu - (k \cdot p)p^\mu][p^2 k^\nu - (k \cdot p)p^\nu]}{k^2(p+k)^2}.
\end{aligned} \tag{A.2}$$

The first integral  $\Pi_{(a)}^{\mu\nu}(p)$  does not receive any noncommutative corrections. Evaluating the integral using standard methods, the same result (II.13) of commutative two-dimensional QED arises. Let us therefore look at the combination  $\tilde{\Pi}^{\mu\nu}(p) = \Pi_{(b)}^{\mu\nu}(p) + \Pi_{(c)}^{\mu\nu}(p) + \Pi_{(d)}^{\mu\nu}(p)$ , which can equivalently be given as

$$\tilde{\Pi}^{\mu\nu} = \sum_{i=0}^2 (1-\xi)^i \tilde{\Pi}_i^{\mu\nu}. \tag{A.3}$$

In what follows, we will compute  $\tilde{\Pi}_i^{\mu\nu}, i = 0, 1, 2$  explicitly. Let us first look at  $\tilde{\Pi}_0^{\mu\nu}$ , which receives contribution from diagrams (b)-(d) in Fig. 2. It is given by

$$\tilde{\Pi}_0^{\mu\nu} = \frac{g^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} \mathcal{R}^{\mu\nu}, \tag{A.4}$$

where

$$\begin{aligned}
\mathcal{R}^{\mu\nu} &\equiv (2d-7)p^2 \delta^{\mu\nu} + (2d-4)k^2 \delta^{\mu\nu} + (4d-6)(p \cdot k) \delta^{\mu\nu} + (6-d)p^\mu p^\nu + (8-4d)k^\mu k^\nu \\
&\quad + (5-2d)p^\mu k^\nu + (3-2d)p^\nu k^\mu.
\end{aligned} \tag{A.5}$$

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<sup>7</sup> We redo the computation here, because at some stages our results turn out to be different from [28].

Using the standard Feynman parametrization method, we get

$$\begin{aligned}\tilde{\Pi}_0^{\mu\nu} &= \frac{g^2}{2} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \ell}{(2\pi)^d} \frac{1 - e^{ip \wedge \ell}}{(\ell^2 + \Delta)^2} \\ &\times \left[ (2d-7)p^2 \delta^{\mu\nu} + (2d-4)(\ell - xp)^2 \delta^{\mu\nu} + (4d-6)p \cdot (\ell - xp) \delta^{\mu\nu} + (6-d)p^\mu p^\nu \right. \\ &\quad \left. + (8-4d)(\ell^\mu \ell^\nu + x^2 p^\mu p^\nu) + (5-2d)(\ell - xp)^\nu p^\mu + (3-2d)(\ell - xp)^\mu p^\nu \right],\end{aligned}\quad (\text{A.6})$$

where  $\ell \equiv k + xp$ , and  $\Delta \equiv x(1-x)p^2$ . The second term in (A.3),  $\tilde{\Pi}_1^{\mu\nu}$  receives contributions from diagrams (b) and (c) in Fig. 2. It is given by

$$\tilde{\Pi}_1^{\mu\nu} = \frac{g^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} \mathcal{S}^{\mu\nu}, \quad (\text{A.7})$$

where

$$\begin{aligned}\mathcal{S}^{\mu\nu} &\equiv \frac{1}{k^2} \left\{ (k^2 + 2k \cdot p)^2 \delta^{\mu\nu} - (k^2 + 2k \cdot p - p^2) k^\mu k^\nu - (k^2 + 3k \cdot p) (k^\mu p^\nu + k^\nu p^\mu) \right\} \\ &+ \frac{1}{(k+p)^2} \left\{ [(k+p)^2 - 2p \cdot (k+p)]^2 \delta^{\mu\nu} - [(k+p)^2 - 2p \cdot (k+p) - p^2] (k+p)^\mu (k+p)^\nu \right. \\ &\quad \left. + [(k+p)^2 - 3p \cdot (k+p)] [p^\mu (k+p)^\nu + p^\nu (k+p)^\mu] \right\} + 2p^\mu p^\nu - 2(k+p)^2 \left( \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right).\end{aligned}\quad (\text{A.8})$$

Introducing the Feynman parameter  $x$  and after a lengthy but straightforward calculation, (A.7) can be given by

$$\begin{aligned}\tilde{\Pi}_1^{\mu\nu} &= 4g^2 \int_0^1 dx (1-x) \int \frac{d^d \ell}{(2\pi)^d} \frac{1 - e^{ip \wedge \ell}}{(\ell^2 + \Delta)^3} \\ &\times \left\{ [(3-2x)x^2 p^4 - (2x+1)\ell^2 p^2 + 4(1-x)(\ell \cdot p)^2] \delta^{\mu\nu} + 2p^2 \ell^\mu \ell^\nu \right. \\ &\quad \left. + (2x-3)(\ell \cdot p)(\ell^\mu p^\nu + \ell^\nu p^\mu) + [(1+2x)\ell^2 - (3-2x)x^2 p^2] p^\mu p^\nu \right\},\end{aligned}\quad (\text{A.9})$$

where  $\ell = k + xp$ . Similarly,  $\tilde{\Pi}_2^{\mu\nu}$ , receiving contribution from diagram (b) in Fig. 2, is given by

$$\tilde{\Pi}_2^{\mu\nu} = \frac{g^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos(p \wedge k)}{k^2(p+k)^2} \mathcal{T}^{\mu\nu}, \quad (\text{A.10})$$

with

$$\mathcal{T}^{\mu\nu} \equiv - \frac{[p^2 k^\mu - (k \cdot p) p^\mu] [p^2 k^\nu - (k \cdot p) p^\nu]}{k^2(p+k)^2}. \quad (\text{A.11})$$

After some straightforward calculation, we arrive at

$$\begin{aligned}\tilde{\Pi}_2^{\mu\nu} &= \frac{g^2}{2} \int_0^1 dx (1-x) \int \frac{d^d \ell}{(2\pi)^d} \frac{1 - e^{ip \wedge \ell}}{(\ell^2 + \Delta)^4} \left\{ (\ell^\mu \ell^\nu + x^2 p^\mu p^\nu) p^4 \right. \\ &\quad \left. - [(\ell \cdot p) \ell^\mu + x^2 p^2 p^\mu] p^2 p^\nu - [(\ell \cdot p) \ell^\nu + x^2 p^2 p^\nu] p^2 p^\mu + [(\ell \cdot p)^2 + x^2 p^4] p^\mu p^\nu \right\}.\end{aligned}\quad (\text{A.12})$$

To proceed, we will use the general relation

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{e^{ip \wedge \ell}}{(\ell^2 + \Delta^2)^n} = \frac{2\pi^{\frac{n}{2}}}{(2\pi)^n} \frac{1}{\Gamma(n)} \frac{1}{(\Delta^2)^{n-\frac{d}{2}}} \left( \frac{|\tilde{p}|\Delta}{2} \right)^{n-\frac{d}{2}} K_{n-\frac{d}{2}}(|\tilde{p}|\Delta), \quad (\text{A.13})$$

and

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell_\mu \ell_\nu e^{ip \wedge \ell}}{(\ell^2 + \Delta^2)^n} = F_n \delta_{\mu\nu} + G_n \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}, \quad (\text{A.14})$$

with

$$\begin{aligned} F_n &= \frac{\pi^{\frac{n}{2}}}{(2\pi)^n} \frac{1}{\Gamma(n)} \frac{1}{(\Delta^2)^{n-1-\frac{d}{2}}} \left( \frac{|\tilde{p}|\Delta}{2} \right)^{n-1-\frac{d}{2}} K_{n-1-\frac{d}{2}}(|\tilde{p}|\Delta) \\ G_n &= \frac{\pi^{\frac{n}{2}}}{(2\pi)^n} \frac{1}{\Gamma(n)} \frac{1}{(\Delta^2)^{n-1-\frac{d}{2}}} \left[ (2n-2-d) \left( \frac{|\tilde{p}|\Delta}{2} \right)^{n-1-\frac{d}{2}} K_{n-1-\frac{d}{2}}(|\tilde{p}|\Delta) \right. \\ &\quad \left. - 2 \left( \frac{|\tilde{p}|\Delta}{2} \right)^{n-\frac{d}{2}} K_{n-\frac{d}{2}}(|\tilde{p}|\Delta) \right], \end{aligned} \quad (\text{A.15})$$

from [28]. Adding the corresponding contributions from diagrams (a)-(d) of Fig. 2, and using  $A+B = \delta_{\mu\nu} \Pi^{\mu\nu}$  arising from (II.17), we arrive at  $A+B$  presented in (II.19)

$$\begin{aligned} A+B &= -\mu^2 - \frac{\mu^2}{4} \int_0^1 dx (x(1-x))^{-1} \left\{ 2(1+2x)(sK_1(s)-1) \right. \\ &\quad \left. - (1-\xi)[6(2x^2-3x+1)(sK_1(s)-1) + (4x^2-6x+1)(s^2K_2(s)-2)] \right. \\ &\quad \left. - \frac{1}{4}(1-\xi)^2(5s^2K_2(s)-2-s^3K_3(s)) \right\}. \end{aligned} \quad (\text{A.16})$$

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